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Enumeration of unrooted orientable maps of arbitrary genus by number of edges and vertices

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ABSTRACT

A *genus-g map* is a 2-cell embedding of a connected graph on a closed, orientable surface of genus g without boundary, that is, a sphere with g handles. Two maps are *equivalent* if they are related by a homeomorphism between their embedding surfaces that takes the vertices, edges and faces of one map into the vertices, edges and faces, respectively, of the other map, and preserves the orientation of the surfaces. A map is *rooted* if a *dart* of the map – half an edge – is distinguished as its *root*. Two rooted maps are equivalent if they are related by a homeomorphism that has the above properties and that also takes the root of one map into the root of the other. By counting maps, rooted or unrooted, we mean counting equivalence classes of those maps.

To count unrooted genus-g maps, we first needed to count rooted maps of every genus up to g. A recursively-defined generating function for counting rooted maps of arbitrary genus by number of faces and vertices was found by Didier Arquès and the second author, and numerical values were obtained for $g \leq 3$. The first two authors jointly extended the solution of this recursion up to $g = 5$. Valery Liskovets used quotient maps and Tutte's enumeration formula for rooted genus-0 maps to count unrooted genus-0 maps by number of edges. The third author and Roman Nedela generalized Liskovets' method to count unrooted maps of genus 1, 2 and 3 by number of edges.

In this paper, we describe the above-mentioned previously-obtained results and present the following new ones. The first author wrote an optimized program in C to extend the solution of the Arquès–Giorgetti recurrence, and thus the enumeration of rooted maps by number of edges and vertices, up to $g = 10$. The second and third authors extended the enumeration of rooted and unrooted maps by number of edges up to genus 11 using tables computed and published on a web site by Ján Karabás, a student of Nedela. Using Liskovets' refinement of the Mednykh–Nedela method and the tables of numbers of rooted maps, the first author counted unrooted maps of genus up to 10 by number of edges and vertices.

1. Introduction: definitions and structure of the article

A *map* is defined topologically as a 2-cell embedding [13] of a connected graph, loops and multiple edges allowed, in a 2-dimensional surface. The *faces* of a map are the connected components of the complement of the graph in the surface. In this article, the surface is assumed to be without boundary and orientable, with an orientation already attributed to it (counter-clockwise, say), so that it is completely described by a non-negative integer g, its *genus*. For short, a map on a

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surface of genus g will be called a *genus-g map*. A *planar map* is a genus-0 map (a map on a sphere) and a *toroidal map* is a genus-1 map (a map on a torus or doughnut). If a map on a surface of genus g has v vertices, e edges and f faces, then by the Euler–Poincaré formula [9, Chap. 9]

$$v - e + f = 2(1 - g). \tag{1}$$

Two maps are *equivalent* if there is an orientation-preserving homeomorphism between their embedding surfaces that takes the vertices, edges and faces of one map into the vertices, edges and faces of the other. A *dart* or *semi-edge* of a map or graph is half an edge. A loop is assumed to be incident twice to the same vertex, so that every edge, whether or not it is a loop, contains two darts. The *degree* of a vertex is the number of darts incident to it. The face incident to a dart d is the face incident to the edge containing d and on the right of an observer on d facing away from the vertex incident to d and the degree of a face is the number of darts incident to it. A *rooted map* is a map with a distinguished dart, its *root*. Two rooted maps are equivalent if there is an orientation-preserving homeomorphism between their embedding surfaces that takes the vertices, edges, faces and the root of one map into the vertices, edges, faces and the root of the other.

A *combinatorial map* is a connected graph with a cyclic order imposed on the darts incident to each vertex, representing the order in which the darts of a (topological) map are encountered during a rotation around the vertex according to the orientation of the embedding surface. The darts incident to a face are encountered by successive application of the following pair of actions: go from the current dart to the dart on the other end of the same edge and then to the next dart incident to the same vertex according to the cyclic order. In this way, the faces of a combinatorial map can be counted, so that its genus can be calculated from (1). Two combinatorial maps are equivalent if they are related by a *map isomorphism* – a graph isomorphism that preserves this cyclic order – with an analogous definition for the equivalence of two rooted combinatorial maps. An *automorphism* of a combinatorial map is a map isomorphism from a map onto itself.

By *enumerating* maps with a given set of properties, whether rooted or not, we mean counting the number of equivalence classes of maps with these properties. It was shown in [13] that each equivalence class of topological maps is uniquely defined by an equivalence class of combinatorial maps; therefore for the purposes of enumeration, the term "map" can be taken to mean "combinatorial map".

To count unrooted maps of genus g, we first needed to be able to count rooted maps of every genus up to g. In Section 2, we summarize the enumeration of rooted maps, both by number of edges and by number of faces and vertices, that we and others have done. In Section 3, we describe the method by which the third author, together with Roman Nedela, enumerated unrooted maps of genus up to 3 by number of edges [21] and a new result: the extension by the second and third authors of these results up to genus 11. In Section 4, we describe another new result, obtained by the first author: adding the number of vertices as a parameter for unrooted maps of genus up to 10. Appendix A contains tables of numbers of unrooted maps of genus 1, 2, 3, 4 and 5 with up to 11 edges, counted by number of edges and vertices. Appendix B contains tables of numbers of unrooted maps of genus 6–11 with up to 60 edges, counted by number of edges. More numerical results can be found in [10].

2. Counting rooted maps

Rooted maps were introduced in [23] because they are easier to count than unrooted maps; this is because only the trivial map automorphism preserves the root [24], so that rooted maps can be counted without considering map automorphisms. In [23], W.T. Tutte found a closed-form formula for counting rooted planar maps with e edges. In [24], he found a recurrence for the number of rooted planar maps given the number of vertices, the number of faces and the degree of the face containing the root. From this recurrence he obtained a three-parameter generating function and then eliminated the parameter for the degree of the root-face to obtain a two-parameter generating function that counts rooted planar maps by number of vertices and faces. In [1], D. Arquès obtained a simpler two-parameter generating function counting the same set of objects.

In [25], the first author generalized the method of [24] to obtain a recurrence for the number of genus-g maps with the vertices labelled $1, 2, \ldots, v$ and with a distinguished dart incident to each vertex, given the number of vertices and the degree of each one; these numbers were then multiplied by the appropriate factor and added over all possible non-increasing sequences of vertex-degrees summing to $2e$ to obtain the number of rooted maps of genus g with e edges and v vertices. A table of these numbers of maps with up to 14 edges appears in [25] (see [30] for a published account of this work and a table of maps with up to 11 edges), but no attempt was made to express these results in terms of generating functions. The algorithm used to solve that recurrence is far from being polynomial-time. The first author later found a polynomial-time algorithm for counting rooted toroidal maps, both by number of edges alone and by number of vertices as well [26], but no explicit formula.

In [5], E. Bender and E. Canfield introduced an improvement on the method of [25]: to count rooted genus-g maps it is sufficient to know the degree of the first $g + 1$ vertices and to distinguish a dart of only the first vertex as the root, thus reducing the number of maps that have to be considered. Using doubly-rooted maps, Arquès [2] obtained a two-parameter generating function that counts rooted toroidal maps by number of vertices and faces. From this result, he obtained a closed-form formula for the number of rooted toroidal maps with e edges and another one for the number of rooted toroidal maps with v vertices and f faces. In [6], Bender and Canfield obtained a generating function for the number of rooted maps of genus 2 and 3 with e edges.

In [11], the second author generalized the results in [2,6] and obtained a general form for the generating function counting rooted maps of any genus $g > 0$ by number of vertices and faces and a recurrence satisfied by a set of multivariate polynomials whose solution determines this generating function. To make these polynomials symmetric in all their variables, he distinguished a dart incident to each of the vertices whose degree is considered, which increases the size of the coefficients but does not increase the number of polynomials that have to be calculated. We note here that in the account of these results published in [3], the recurrence is missing a term; this omission was corrected in [29]. Programming in Maple, he solved the recurrence explicitly for $g = 2$ and $g = 3$ (these results are published in [3]) and also computed the generating function that counts rooted maps of genus 4 by number of edges. This result was recently included in [20], where it was used by the second and third authors to count both rooted and unrooted maps of genus 4 by number of edges. This was as far as the program written in this version of Maple was capable of carrying the calculations.

Recently, using a newer version of Maple and a more powerful computer, he extended the solution of his recurrence by number of vertices and faces up to genus 5. This is as far as the program written in the newer version of Maple was capable of carrying the calculations; it could have been optimized, but that was not its objective. To improve the computational efficiency of the calculations, the first author programmed mainly in *C*, using the library CLN (Class Library for Numbers) to handle big numbers – see the web site [15] – and the set of tools Xcode to run CLN – see the web site [33]. He optimized the solution to the recurrence in [11] and thus extended the enumeration by number of vertices and faces, as well as by number of edges, to genus 6. Although each author used a different algorithm and a different programming language, we both obtained the same answers, and the numbers of rooted maps we calculated agree with the tables in [25], providing evidence of the correctness of our results. These results are presented in [29].

More recently, the first author, using a more powerful computer, extended the enumeration of rooted maps by number of vertices and faces to genus 10 and the second author solved the one-parameter version of his recurrence, which counts rooted maps of genus g by number of edges, up to genus 11. Using these results, the third author, programming in Mathematica, counted unrooted maps of genus up to 11 by number of edges (see Section 3) and the first author, programming in *C*, counted unrooted maps of genus up to 10 by number of edges and vertices (see Section 4). The results described in this paragraph are new.

3. Counting unrooted maps by number of edges

The method used in [21] for counting unrooted maps by number of edges is a generalization of the method used by Valery Liskovets [16,17] for counting unrooted planar maps by number of edges. It uses the so-called Burnside's Lemma [22], which states that the number of orbits of a finite permutation group A acting on a set X is equal to the sum over all the elements a of A of the number of objects in X fixed by a divided by the cardinality of A. The set X is the set of darts of a map M with labelled darts. If M has E edges, it has $2E$ darts, so that $\#(A) = (2E)!$. A permutation a of the darts of M fixes M if a is an automorphism of the unlabelled version of M. However, the action of an automorphism of a map is completely determined by its action on a single dart [24]; so each rooted map can be dart-labelled in $(2E-1)!$ ways with the root getting the label 1. Applying this observation to Burnside's Lemma yields the following formula:

$$N^{+}(E) = \frac{1}{2E}\sum_{a} \mathrm{fix}(a, E), \tag{2}$$

where $N^{+}(E)$ is the number of unrooted maps with E edges, a runs over all the permutations of the darts of a rooted map with E edges and $\mathrm{fix}(a, E)$ is the number of pairs (a, M), where M is a rooted map with E edges and a is an automorphism of the unrooted version of M.

A map automorphism is *regular* on its set of darts, that is, it is a permutation consisting of independent cycles of the same length [24]. One can thus consider only periodic orientation-preserving homeomorphisms of the embedding surface and call them automorphisms of the surface.

For the sphere, the non-trivial automorphisms are just rotations [4]. A rooted map M with an automorphism a of period L can thus be drawn on the sphere in such a way that the automorphism can be realized topologically by a rotation of period L. If the sphere is then sliced into L lunes and the lune containing the root inflated to a sphere, it will contain a rooted map with only $2E/L$ darts, the *quotient map* m of M under the automorphism a. The axis of rotation will pass through two distinct *cells* (vertices, edges or faces) of M, which we call its *poles*. If a pole is a *non-edge* (a face or a vertex), then it will be a non-edge of m. If a pole is an edge, then it will be a *dangling* semi-edge of m — a semi-edge that is not part of a normal edge. If at least one pole is an edge, then L must be 2, there is only one such automorphism, and m will have 1 or 2 dangling semi-edges. Once the axis of rotation has been chosen, the number of rotations of period L is $\phi(L)$, where ϕ is the Euler totient function. In [17,16], an elegant formula was obtained for the number of unrooted planar maps with E edges by counting the rooted maps, possibly with one or two dangling semi-edges, that can be a quotient map of some rooted map with E edges under an automorphism of period L, multiplying by the number of ways of distributing the poles among the non-edges and dangling semi-edges of the quotient map and the number of automorphisms of period L of a sphere, summing over L and dividing by $2E$.

In [21], this method was generalized to surfaces of arbitrary orientable genus. We illustrate this method on the torus.

We first represent a torus as a square with opposite sides identified in pairs, so that the four corners of the square represent a single point on the torus. If the square is rotated by 180° ($L = 2$), there are four points on the torus that are fixed: the centre of the square, the point represented by the four corners and each of the two points represented by the middle of a pair of opposite sides. This automorphism thus has four orbits of length less than the period. Such an orbit is called a *branch point*, and the *branch index* of a branch point is L divided by the orbit length, in this case 2. This is the only automorphism of this sort. If instead the square is rotated by 90° ($L = 4$), the centre and the point represented by the four corners are fixed (two branch points of branch index 4) and the middles of both pairs of opposite sides are in a single orbit of length 2 (one branch point of branch index 2). There are two such automorphisms.

Next we represent a torus as a hexagon with opposite sides identified in pairs, so that each triplet of non-adjacent corners represents a single point. If the hexagon is rotated by 120° ($L = 3$), the centre is fixed and so is each triplet of non-adjacent corners (three branch points of branch index 3); there are two such automorphisms. If instead the hexagon is rotated by 60° ($L = 6$), the centre is fixed (a branch point of branch index 6), there is an orbit of length 2 (a branch point of branch index 3) containing the two triplets of non-adjacent corners and an orbit of length 3 (a branch point of branch index 2) containing the middle of all three pairs of opposite sides. There are two such automorphisms. For all these automorphisms, the quotient map is of genus 0.

There is also an infinite family of automorphisms with no branch points generated by two independent rotations: rotating the torus around the middle of the hole like the tube inside a tyre on a spinning bicycle wheel and twisting it so that the valve no longer points to the centre of the wheel. The number of such automorphisms of period L is $\phi_2(L)$ where $\phi_k(L)$ is the kth Jordan function of L ($k \geq 1$). For all these automorphisms, the quotient map is of genus 1.

In [21], the quotient space of a surface of genus G under an automorphism of some period L is called a *G-admissible orbifold* O, and it is characterized by its *signature*: its genus g and its branch indices $m_1, \ldots, m_r$, where $1 < m_1 \leq \cdots \leq m_r$. Each such automorphism of period L corresponds to an (order-preserving) epimorphism: a surjection, with a torsion-free kernel, of the automorphism group of the surface of genus G onto the cyclic group Z_L of order L. In [21], the number of epimorphisms is given as a function of L and the signature $[g; m_1, \ldots, m_r]$ by the formula

$$Epi_0(\pi_1(O), Z_L) = m^{2g}\phi_{2g}(L/m)E(m_1, \ldots, m_r), \tag{3}$$

where $m = \mathrm{lcm}(m_1, \ldots, m_r)$, m divides L and

$$E(m_1, \ldots, m_r) = \frac{1}{m}\sum_{k=1}^{m}\prod_{i=1}^{r}\Phi(k, m_i), \tag{4}$$

where

$$\Phi(k, n) = \frac{\phi(n)}{\phi(n/\gcd(k, n))}\mu(n/\gcd(k, n))$$

is the von Sterneck function (see [18] for a discussion of the relation between this function and a certain trigonometric sum of Ramanujan). Here μ is the Möbius function, and in [21] the formula involving μ is used to express the Jordan function.

To avoid having to work with too many non-decreasing sequences of integers $(m_1, \ldots, m_r)$, the authors of [21] used the criterion in [12] for the existence of a G-admissible orbifold of period L and signature $[g; m_1, \ldots, m_r]$:

H1: the Riemann–Hurwitz equation

$$2 - 2G = L\left(2 - 2g - \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right), \tag{5}$$

H2: $m = \mathrm{lcm}(m_1, \ldots, m_r)$ divides L and $m = L$ if $g = 0$,
H3: $\mathrm{lcm}(m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_r) = m$ for each $i = 1, 2, \ldots, r$,
H4: if m is even, then the number of those m_i that are divisible by the highest power of 2 that divides m is even,
H5: if $G \geq 2$, then $r \neq 1$ and $r \geq 3$ for $g = 0$, if $G = 1$ then $r \in \{0, 3, 4\}$, if $G = 0$ then $r = 2$.

They also bounded L using the Wiman theorem [7, page 131]: if $G > 1$, then $1 \leq L \leq 4G + 2$.

They then expressed the orbifold signature in terms of the number q_i of branch points with branch index i for every i from 2 to L and obtained the following formula for the number of rooted maps $\nu_O(d)$ with d darts on an orbifold $O = O[g; 2^{q_2}, \ldots, L^{q_L}]$:

$$\nu_O(d) = \sum_{s=0}^{q_2}\binom{d}{s}\binom{(d-s)/2 + 2 - 2g}{q_2 - s, q_3, \ldots, q_L}N_g((d-s)/2), \tag{6}$$

where $N_g(n)$ is the number of rooted maps of genus g with n edges (0 if n is not an integer). Here s is the number of dangling semi-edges in the quotient map m, all of which must be in orbits of length $L/2$ so that they represent normal edges in the original map M. Given a rooted map with $d-s$ darts, dangling semi-edges can be inserted into the slots between two adjacent darts according to the rotation about their common incident vertex, and since more than one dangling semi-edge can be inserted into the same slot, the number of ways to insert s dangling semi-edges is $\binom{d-1}{s}$. In addition, there are d ways to root the map once the dangling semi-edges have been inserted and only $d-s$ ways to root it before the insertion. Multiplying the number of insertions by the ratio of the number of ways to root the map with and without the dangling semi-edge gives the first factor in the right side of (6). The multinomial coefficient in (6) is the number of ways in which the branch points with the various branch indices can be distributed among the non-edges of the quotient map; the "numerator" of the multinomial coefficient is the number of non-edges and is given by (1).

For each period L, if the original map has E edges, then the quotient map will have $2E/L$ darts. Substituting $2E/L$ for d in (6), multiplying by the number of epimorphisms of a given orbifold (3) and adding over all the G-admissible orbifolds with period L and then over all the periods L that divide E and finally dividing the sum by $2E$, as in (2), they obtained a formula for the number of unrooted maps of genus G with E edges:

$$\frac{1}{2E}\sum_{L|E}\sum_{O} Epi_0(\pi_1(O), Z_L)\nu_O(2E/L), \tag{7}$$

where O runs over all the G-admissible orbifolds with period L and its signature is expressed as $[g; m_1, \ldots, m_r]$ when substituting into (3) and as $[g; 2^{q_2}, \ldots, L^{q_L}]$ when substituting into (6).

Of course, to get explicit numbers, they needed to know the number of rooted maps of genus up to G and the set of G-admissible orbifold signatures, along with the number of epimorphisms for each one. The former numbers were calculated from the formulae in [2] for $G = 1$ and in [6] for $G = 2$ and 3; the latter, for $G \leq 4$, were available from various sources. Once the second author provided the third one with a generating function for the number of rooted maps of genus 4 with E edges, the number of unrooted maps of genus 4 with E edges could be counted [20]. When the second author extended his enumeration of rooted maps with E edges up to genus 11, the third author needed the G-admissible orbifolds and their number of epimorphisms up to $G = 11$. These were provided by Ján Karabás, a student of Nedela, who computed them (programming in Magma) from Harvey's condition and formulae (3) and (4) for G up to 100 and made them available on his web site [14]. With these results and the generating functions provided by the second author, the third author extended the enumeration of unrooted maps with E edges up to genus 11.

4. Counting unrooted maps by number of edges and vertices

In [27], the first author used the method of [16,17] to obtain a formula for the number of unrooted planar maps with E edges and V vertices. It is to be noted here that Nicholas Wormald counted planar maps by number of edges and vertices up to not only orientation-preserving isomorphism but also orientation-reversing isomorphism [31,32]. The formula in [16,17] and the one in [27] have smaller asymptotic computational complexities than the respective algorithms in Wormald's papers restricted to counting unrooted planar maps up to orientation-preserving isomorphism. Liskovets suggested a way in which his method could be used to count unrooted planar maps up to both kinds of isomorphism [19]; finding an algorithm more efficient than Wormald's using the results in [19] is an interesting open problem.

The basic idea used in [27] is to distribute the branch points (poles, in the case of planar maps) among the vertices, faces and dangling semi-edges of the quotient map instead of just among the non-edges and the dangling semi-edges. We now apply this idea to counting unrooted maps of genus G by number of edges and vertices. Suppose that the quotient map is of genus g and has v vertices, f faces and s dangling semi-edges. Then the number e of normal edges can be calculated from (1) and the number d of darts is $2e+s$. Suppose also that there are v_k branch points of orbit length k (orbit length $= L$ divided by branch index) that are on a vertex and f_j branch points of orbit length j that are in a face. We denote by v_L and f_L the number of vertices and faces, respectively, that do not contain a branch point. The original map will have k vertices for every vertex in a branch point of orbit length k, L vertices for every vertex not in a branch point, j faces for every face in a branch point of orbit length j and L faces for every face not in a branch point. The total numbers V of vertices and F of faces in the original map are given by the formulae

$$V = \sum_{k=1}^{L} k v_k \tag{8}$$

and

$$F = \sum_{j=1}^{L} j f_j, \tag{9}$$

and the total number E of edges is equal to $Ld/2$.

The binomial coefficient in (6) does not change – it still represents the number of ways to insert s dangling semi-edges into a rooted map with $m-s$ darts and the ratio of the number of rootings – but the multinomial coefficient in (6) must be modified. The number of ways to distribute the branch points among the vertices and faces is

$$\binom{v}{v_1, \ldots, v_L}\binom{f}{f_1, \ldots, f_L}. \tag{10}$$

For this number to be positive, the sum of all the numbers v_k cannot exceed v and the sum of all the numbers f_j cannot exceed f; so v and f each start at its respective sum and increases by 1 until the number E of edges in the original map exceeds a user-defined maximum. With each increase of v or f, (10) gets updated using a single multiplication and division. This number, which replaces the multinomial coefficient in (6), must be computed for all sets of non-negative integers such that for each k, $v_k + f_k$ is equal to the total number of branch points of orbit length k that are not distributed to dangling semi-edges.

Once (6), modified as described above, is multiplied by the number of epimorphisms of the current period and orbifold signature, we get the contribution of that period, signature and the numbers v_k and f_j to $2E$ times the number of unrooted maps of genus G with E edges and V vertices. This contribution is added to the appropriate element of a two-dimensional array, initially 0, and when all the contributions have been tallied, for each E and V the corresponding array element is divided by $2E$.

The first author programmed this calculation in C to obtain tables of numbers of unrooted maps of genus up to 10, counted by number of edges and vertices. Instead of taking the set of orbifold signatures and the number of epimorphisms for each one from [14], he recalculated them, not from Harvey's condition and formula (4) but from a refinement of these two results recently published by Liskovets [18], except that orbit lengths were used instead of branch indices.

Here is a summary of the results taken from [18]. Given an r-tuple $(m_1, \ldots, m_r)$ of integers, each greater than 1, let m be their least common multiple. For every prime p that divides m, let $a(p)$ be the exponent of p in the prime power factorization of m, and for each index $j = 1, 2, \ldots, r$, let $a_j(p)$ be the corresponding exponent for m_j. Now let $s(p)$ be the number of indices j such that $a_j(p) = a(p)$. Then Harvey's condition is equivalent to the following:

E1: the Riemann–Hurwitz equation (5),
E2: m divides L,
E3: either $g \neq 0$ or $L \leq m$,
E4: $s(p) \neq 1$ for every odd prime p that divides m,
E5: if m is even, then $s(2)$ is also even.

The formula in [18] that replaces (4) is

$$E(m_1, \ldots, m_r) = \prod_{\substack{p \text{ prime} \\ p \text{ divides } m}} (p-1)^{r(p)-s(p)+1} p^{v(p)} h_{s(p)}(p), \tag{11}$$

where $r(p)$ is the number of those m_j that are divisible by p,

$$v(p) = \sum_{\substack{j=1,\ldots,r \\ a_j(p)\geq 1}} \left(a_j(p) - 1\right) - a + 1,$$

and $h_s(x) = \left((x-1)^{s-1} + (-1)^s\right)/x$. He also showed that $r \leq 2G+2$ and used the multiplicative formula for the kth Jordan function

$$\phi_k(n) = n^k \prod_{p|n \text{ prime}} \left(1 - p^{-k}\right), \tag{12}$$

and revealed its important role in unlabelled enumeration (for diverse types of groups, maps and some other algebraic and topological objects).

Formula (12) was used by the first author so that, in particular, a table of values of the second Jordan function, which is needed to count unrooted toroidal maps, could be evaluated using a sieve-like algorithm.

The numbers calculated by all three authors and by Karabás agree with each other and with the numbers of unrooted maps of genus g with up to 6 edges in [28] and the number of unrooted maps of genus g with one face, one vertex and $2g$ edges in [8]. Source codes producing most of the new results are included in release 0.3 of the MAP project [10]; more source codes will be included there shortly.

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Appendix A. Numbers of unrooted maps of positive genus with E edges and V vertices, up to genus 5

E	V	Genus 1	Genus 2	Genus 3	Genus 4	Genus 5
2	1	1				
2	Sum	1				
3	1	3				
3	2	3				
3	Sum	6				
4	1	11	4			
4	2	24				
4	3	11				
4	Sum	46	4			
5	1	46	53			
5	2	180	53			
5	3	180				
5	4	46				
5	Sum	452	106			
6	1	204	553	131		
6	2	1198	1276			
6	3	2048	553			
6	4	1198				
6	5	204				
6	Sum	4852	2382	131		
7	1	878	4758	4079		
7	2	7212	18582	4079		
7	3	18396	18582			
7	4	18396	4758			
7	5	7212				
7	6	878				
7	Sum	52972	46680	8158		
8	1	3799	35778	73282	14118	
8	2	40776	205867	167047		
8	3	142727	347558	73282		
8	4	212443	205867			
8	5	142727	35778			
8	6	40776				
8	7	3799				
8	Sum	587047	830848	313611	14118	
9	1	16304	244246	970398	684723	
9	2	219520	1910756	3693031	684723	
9	3	999232	4747430	3693031		
9	4	2040348	4747430	970398		
9	5	2040348	1910756			
9	6	999232	244246			
9	7	219520				
9	8	16304				
9	Sum	6550808	13804864	9326858	1369446	
10	1	69486	1552834	10556722	17586433	2976853
10	2	1139075	15680071	58591595	39630698	
10	3	6488604	52969260	97799324	17586433	
10	4	17227356	77948670	58591595		
10	5	23634214	52969260	10556722		
10	6	17227356	15680071			
10	7	6488604	1552834			
10	8	1139075				
10	9	69486				
10	Sum	73483256	218353000	236095958	74803564	2976853

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E	V	Genus 1	Genus 2	Genus 3	Genus 4	Genus 5
11	1	294350	9349284	99944546	319763792	195644427
11	2	5741220	117450580	748976684	1192082898	195644427
11	3	39779852	512308352	1823736772	1192082898	
11	4	132209016	1025303224	1823736772	319763792	
11	5	235876296	1025303224	748976684		
11	6	235876296	512308352	99944546		
11	7	132209016	117450580			
11	8	39779852	9349284			
11	9	5741220				
11	10	294350				
11	Sum	827801468	3328822880	5345316004	3023693380	391288854

Appendix B. Numbers of unrooted maps of higher genus with E edges

B.1. Genus 6

E	
12	1013582110
13	169857459866
14	15202559941266
15	964949198577434
16	48695382495936280
17	2077641325234343160
18	77856954571873092792
19	2629247239663140056192
20	81498694882848919958250
21	2350709723279734060249864
22	63759822591543968176310334
23	1639827731755812039740785472
24	40257891518670970262942165090
25	948585942890219490237384591088
26	21549766011971541181992572512626
27	473806340178120118849640022425900
28	10114720083222432999084227398358452
29	210233889853395266879088551176579820
30	4264663911331018717169989589125057766
31	84606282019529240955547128954226420384
32	1644547580696596356283284678184363982368
33	31369856210493616467385101426972150290704
34	588054415122647539759866591429485551734236
35	10847016096589410426790059764172093353154416
36	197097727803842741477409520879231539799898234
37	3531605493381525657637657053383598337564471784
38	62456722605458947959862713526990407373193996604
39	1091093478677654631802421386606786951584480525820
40	18842851456763416437596362136877659819596255517766
41	321906777466364975858545625382629563414633571361920
42	5443567218704833627180745327314645286439369134196300
43	91170702206976820724558766964165889013817864914802048
44	1513119246696008312312381749459638849061814170401885504
45	24896991961471615935669206635983903605294711880681920876
46	406321591324580394279291467579787049007393238165165809186
47	6579926002830861449131941416455816910219756843076190654176
48	105770802758308360649883275547456279398636892674967281404934
49	1688332360113800227344172962437147367130483913462644689293064
50	26769503180911409295212289975560590426597868605628610383956316
51	421741131414261390365250055300968461714036961037663191708727384
52	6603877619095479394041465680864389147330704238256026354851992632
53	102805290916811822384730495178454711414523632854911272445698043800
54	1591489740513294120349552678423022034673592147322610861643058070424
55	24505629320849823872299106574761920704725221763646566903644837510176
56	375402333685850434731056323947407640723415592863505834611830289045386
57	5722513985120073082568705723415079278905517797475281261492732739140352
58	86820031023931732418825931167760655797410467785357932444302773736605636
59	1311220380897797897948113538662760855936410788676319733879654819445246848
60	19716467460606829090372643488576345418912032868813407075541441944319444660

B.2. Genus 7

E	
14	508233789579
15	104295987346126
16	11269592389125547
17	852994611088758224
18	50777879440443305426
19	2531246455428148382456
20	109880399953287962099588
21	4265557888300762164284822
22	150940131172496245801920542
23	4938911033961317567088755908
24	151101665358744941452325232448
25	4360483754199984715074014714944
26	119539870847234909092111483374240
27	3131407562025233931111286848825336
28	78764171967356631126862089521296434
29	1910128931163599091900790554101482304
30	44818818312734814243488872935852405822
31	1020534791963767399897494242497438137500
32	22609802197454151252784951086581705441804
33	488489452361574309575227903490929953320896
34	10312666420908045250643751962941571359203560
35	213113473236984907441203120755930069491239740
36	4317735932425414748542185285396388703594953560
37	85884406967080490368421278510825820581519684736
38	1679308578558910438139871625438667532346968963740
39	32314321530844683889275577946292854231258683690032
40	612563302802467478758385690748927865422533574009368
41	11449854422129831786234844778642810255254513953830912
42	211206521123618587279455077458501243649615915721982712
43	3847745282639877595262508094832099007998518363763876992
44	69279034324099303200057755146684590451082091694734348620
45	1233597619651933959307004683816684283533045891623491163904
46	21735947600141356085168183739891673529232656689639562077946
47	379187520557371834560024177310536508911348526208934123623372
48	6552678629454458018083440027886111945159116999979199234063568
49	112221426194934471641475599278717354989485802664710442898240588
50	1905516877667653414169804300202582531747881777142351783275577816
51	32092580693685357648296511857643351044032531659750699730280245280
52	536307623843649192548743905069198235984856481229513763879337837432
53	8895952867062286958560545481155282062351295593023676879206009160448
54	146514913226747465594293697888164822850233324177712596154726289940308
55	2396705045433193868029416004089852954492279050489369664397728918352632
56	38950669045063840205165530186000835841869243929285467693660480026530002
57	629070510133241539461531898472688752514114604942462648906808978092896576
58	10098973904575410163908560253038101598369504586131263340098555697266949376
59	161119502861865858210422826052146307673330312268524649958957488487237721977 6
60	2558700598013007418353936748573768749516970000812621649148379229732358938604

B.3. Genus 8

E	
16	352755124921122
17	86116887841273186
18	10954787876407932816
19	967146341367928365308
20	66599326875830666763353
21	3811863659211606517416928
22	188710867264106506704457217
23	8303453286421604392505856232
24	331175730212422476849562734689
25	12151338155016475304016716988472
26	414905363192585645372577162679456
27	13304794924310393122595064799408284
28	403671763868062607417900669369211420
29	11659290153327679072302859916232402352
30	322230358567237591792303701750192558742

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E	
31	85584693269756201640564720472435949764 16
32	2192664556536802378562273944785757450897 24
33	5436060676593730958693660086222459737282140
34	13077966389352595847636260998697914709216 5344
35	306053362677196073373166580907006376186080 1112
36	6982103232738848877493787855966901589015747 5542
37	155572320366292930518892558697416141942826361 0056
38	3391326965744498606108026351472128607955729661 7062
39	72436108348870848221583564832745989054647421124 0656
40	151802435397376944732924144903599076362676322401 35034
41	3125163460356062114758963796660434957730024053159 79104
42	6327306097012246326331832817999842560054127217223 834138
43	12611147780851304517225717578007575240133303449911 5888832
44	247673035452668761655935671180148680968047067460876 5943832
45	4796843052064938758411225118282963127813101299499585 6133336
46	91689333746479785974412878901660944644233376570288745 9345522
47	173091353351704127427742043595159213495255203385056610 59094784
48	3229294544842365810415413402389081349249031605567374367 56057294
49	59576697060158187293411138486914432595014147484418309987 22662812
50	108748399268898628485334709133537214375667812333089796447 360204456
51	1965036409046907543763854944187831846798163665456356089975 282774792
52	35166347734797508089843922460068260354581725371363276734959 236285702
53	623571248297520083194595893195183870281006183589651263757299 744557176
54	1096041859316803163990491600647525398152011616547029164091393 7108038840
55	19103747797805588088048882397062606031568742366977282142386038 7102049264
56	330306872110045485951205337049574753525836857561805785663783042 78358367 18
57	5667238338681274283936315558082875221840122915695273577507422461 1969302176
58	96520211568477652482807991441185609749460646645769288993422216836 4104536544
59	163225366332041135042058427955218808421191594421056325283098964034 94502707936
60	2741584170774631592706818271304386372416013534690598352603712346773 98728075596

B.4. Genus 9

E	
18	324039613564554401
19	9207573836887674871 0
20	135249208706674468194 90
21	1368913666872922446728390
22	10736756560641800896457633 8
23	695784795298332744124844590 8
24	38788352543237676935391507557 1
25	191229302587036008712069125213 52
26	8506286209043276000857979062497 2
27	346607041976957488176216417905442 74
28	1309108107013455070987022922625639404
29	46264804238544384949037786444224817808
30	1541697807546582972003824248495197791040
31	48750868072778726322358572348682526118512
32	1470674413023265109445369421391649380002680
33	42517503351634513505034746317190444152718100
34	1182554513408384863791311626413295059577278918
35	31749282138634338483529022868508621494567389972
36	825231236977862967585824445839359217937064821108
37	20819024293510525205215523215789649987890445184456
38	510941761751850606717527876906671617663779333780844
39	12223107477177674400264266062377906069697979822194040
40	285541868002971002248385267738691384860486575614655930
41	6524298746718468036961505245270945756563322819681846128
42	146016496950422981490743699401830107090434393352189672088
43	3205100243681845379234152052885426378137651515891765171872
44	69082422837576323748490370498937956412557984865132478615112
45	1463686170010639361057252373771834698549221169729345655198408
46	30514772208717493576469920449768771174170533288604118170019744
47	626534680948481435631458643567390712835194802349271183244114880
48	12679782364017782674092807512675141999037547396468983220359263690
49	253128099639900064366770808217646608887339168513200632299729809536
50	4988101508739129865931324741353295013014037789398983541242133063274

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E	
51	970907825851761871648108881043014455954454515208246555494901 70227036
52	18677948819208514090137146698133989295707826579572256382264653933965 94
53	355329068251874001808973737647255982874832280235247097399631474679008 24
54	6688174709351012066756225226197576051728907838044373149169796377820782 24
55	12461471446531684774904990821780893677992830405652517800783987070738641464
56	229938342419540493658546637308943235903268707813453078105222755166062631042
57	4203553088855004439823661022790914678911070756413216064709810245818335677848
58	76165058225130684780460355694243328216035944559435838429817712677431407832504
59	1368325583133183046618916921496345500865278182180103279804283183507142182568464
60	24381927970987671914328425225463954792390612853240756156936856158125712891891616

B.5. Genus 10

E	
20	380751174738424280720
21	123800701289478148878890
22	20679270860399513431761798
23	2366561529248819497695971912
24	208773430159079852919281433050
25	15143758135416335992767275804168
26	940742579115450773885994408739386
27	51467173203510450132338681924301520
28	2530727117428428438308400215659985178
29	113582174788226730717481987854720696072
30	4709316234749504844812923662857896060560
31	182128999558262362218378458897222822714976
32	6622104249432921001884424488241485750201396
33	227849526460717686998848846056139571941630156
34	7459736467277700057179890548715263005370985966
35	233482060481380107243093490746078128691962484200
36	7014286217890225417226739389328779976760911466074
37	202967156960523834442064922021645002827316885056580
38	5674152031602766060304354036097476022844619287183262
39	153663257902582984516387992836133941211078907688181040
40	4040729627010699761174194256962911776301566632775726888
41	103391155715507965146048772593271871266327516464824352504
42	2579033263971157145487348856673139205312786067962699647468
43	62822199990466322705619399381921717695865416449284600141888
44	1496623303980692304850868640619508553932726722480291964131378
45	34918273872805775937997628975051607979716627095967926569132624
46	798871188192595035461977213060305295013147450083519319918229456
47	17942338645589558763341153213859094070852995759578925931992760192
48	396014981135504665607563643829159732069278357835189600774648178330
49	8597826453984990401769353020057262773367563230373577715120815702696
50	183776868477958268957097006666573217111926165527774738021603558884882
51	3870514935637649537074593083334038717500459439699681147177073379227696
52	80379649130474183022619035120111771942765468073398294246126458356688132
53	1647109543302622956745179276645139173475770812714158077554362371513937260
54	33325421527070646193720640266033519119443406087650443525404742379766570234
55	666137190644517636452053994116097516227059690127420162286598104500304545296
56	13162148460007486373292224203811920901927141490042924822494272697369373000 7964
57	257210649320820287037920746866087591701133038030003634592068764667505123815192
58	4973468382546634810087949457145600917990750720257129230796070135285374229922620
59	95199294862031929685183920826117729279769278511048148305583492408592999760084672
60	1804660055394720947762511220970566674142748688112910407532134558008987230195469932

B.6. Genus 11

E	
22	557175918657122229139987
23	20449371298793674921492 6846
24	383638316030047609979414 51373
25	4908182967303346419185314418168
26	48197457136005621468868471007497 0
27	3876031852135021704915103571470114 0
28	265953095802904044006165287561876996 0
29	16015017225807778294353118598479052486 4
30	863942125649013366407770895640623304915 4

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E	
31	4240918185123456744121211038334440005000072
32	19176410851450405915788809062714452752159892
33	806625895990318779950685088207083343604955762
34	31817411237162562288828285224093499379441512120
35	1184807391310332689716680947455513255790389290320
36	41885799474999590506447644008850335443744496923588
37	1412559751184496833017754842551197512566051180065216
38	45631252235198098800530246769985961593525853504254454
39	1417070681001040858395390832516223158558417457844307028
40	42438214110152736925280923319475095543671820358249697424
41	1229021184230094455942277864231633270595496956449731609712
42	34503380124737380239933765526317830486019323729647759714572
43	941046464139287956413663018360258975839544200706886574362104
44	24983624699698641859031132153385795714893538310726589720564358
45	646780717532345371076921014969558255750393622971486781062282800
46	16353251186472052031914630447464146777966525521734823254883910160
47	404408449730657155398572855369583115358781362962424185180303412128
48	9794280045681508734609085288588870345536526164041734329181258554504
49	232583212389342421535246625847758015797902671000845189507320885658240
50	5421396376113795207505238312634504369601219184890432282173411732863236
51	124166740805241613594725331101009618378024129461979652842202137942890944
52	2796793054707355425614091316721276875151443456382221547670793695947613440
53	62007587160781389196040709642714374483018986086241637456631077536890110848
54	1354251767851339437482957063532386364671344727650927212179563513054726022842
55	29156782683597269863768317268904333898653733169100660122962416735936186546368
56	619238964442792446512922422613940681550079183421149506636393285339400613893648
57	12981587697255782012366544649520592809020598722635915269799378455970290994258864
58	268783109183797840139462084777605469450539186928227579103927797364025867403382028
59	5499429000887647647478144455274508727207589066841448266040991298557559140229329416
60	111248994115959865774034923220182156399517231901932153556694553419555174647929712576

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